forces. This means that at zero frequency there should be no dependence of $\mathbf{P}$ on $\mathbf{B}_0$, and a more realistic form is

$$\frac{1}{\epsilon_0}\mathbf{P} = \chi_0\mathbf{E} + \chi_1 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}_0 + \chi_2'(\mathbf{B}_0 \cdot \mathbf{B}_0) \frac{\partial^2 \mathbf{E}}{\partial t^2} + \chi_3'\left(\frac{\partial^2 \mathbf{E}}{\partial t^2} \cdot \mathbf{B}_0\right)\mathbf{B}_0 \qquad (6.149)$$

where we have exhibited only the lowest order time derivatives for each power of $\mathbf{B}_0$. At optical frequencies this equation permits an understanding of the gyrotropic behavior of waves in an isotropic medium in a constant magnetic field.*

Another example, the Hall effect, is left to the problems. It, as well as thermogalvanomagnetic effects and the existence of magnetic structure in solids, are discussed in *Landau and Lifshitz* (*op. cit.*).

In certain circumstances the constraints of space-time symmetries must be relaxed in constitutive relations. For example, the optical rotatory power of chiral molecules is described phenomenologically by the constitutive relations, $\mathbf{P} = \epsilon_0\chi_0\mathbf{E} + \xi\partial\mathbf{B}/\partial t$ and $\mu_0\mathbf{M} = \chi_0'\mathbf{B} + \xi'\partial\mathbf{E}/\partial t$. The added terms involve pseudoscalar quantities ξ and ξ' that reflect the underlying lack of parity symmetry for chiral substances. (Quantum mechanically, nonvanishing ξ or ξ' requires both electric and magnetic dipole operators to have nonvanishing matrix elements between the same pair of states, something that cannot occur for states of definite parity.)

6.11 On the Question of Magnetic Monopoles

At the present time (1998) there is no experimental evidence for the existence of magnetic charges or monopoles. But chiefly because of an early, brilliant theoretical argument of Dirac,† the search for monopoles is renewed whenever a new energy region is opened up in high-energy physics or a new source of matter, such as rocks from the moon, becomes available. Dirac's argument, outlined below, is that the mere existence of one magnetic monopole in the universe would offer an explanation of the discrete nature of electric charge. Since the quantization of charge is one of the most profound mysteries of the physical world, Dirac's idea has great appeal. The history of the theoretical ideas and experimental searches up to 1990 are described in the resource letter of Goldhaber and Trower.‡ Some other references appear at the end of the chapter.

There are some necessary preliminaries before examining Dirac's argument. One question that arises is whether it is possible to tell that particles have magnetic as well as electric charge. Let us suppose that there exist magnetic charge and current densities, ρ_m and $\mathbf{J}_m$, in addition to the electric densities, ρ_e and $\mathbf{J}_e$. The Maxwell equations would then be

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_e, \qquad & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_e \\ \nabla \cdot \mathbf{B} &= \rho_m, \qquad & -\nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m \end{aligned} \qquad (6.150)$$

*See Landau and Lifshitz, *Electrodynamics of Continuous Media*, p. 334, Problem 3, p. 337.

†P. A. M. Dirac, *Proc. R. Soc. London* **A133**, 60 (1931); *Phys. Rev.* **74**, 817 (1948).

‡A. S. Goldhaber and W. P. Trower, Resource Letter MM-1: Magnetic Monopoles, *Am. J. Phys.* **58**, 429–439 (1990).

The magnetic densities are assumed to satisfy the same form of the continuity equation as the electric densities. It appears from these equations that the existence of magnetic charge and current would have observable electromagnetic consequences. Consider, however, the following *duality transformation**:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}' \cos \xi + Z_0 \mathbf{H}' \sin \xi, & Z_0 \mathbf{D} &= Z_0 \mathbf{D}' \cos \xi + \mathbf{B}' \sin \xi \\ Z_0 \mathbf{H} &= -\mathbf{E}' \sin \xi + Z_0 \mathbf{H}' \cos \xi, & \mathbf{B} &= -Z_0 \mathbf{D}' \sin \xi + \mathbf{B}' \cos \xi \end{aligned} \tag{6.151}$$

For a real (pseudoscalar) angle ξ, such a transformation leaves quadratic forms such as $\mathbf{E} \times \mathbf{H}$, $(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$, and the components of the Maxwell stress tensor $T_{\alpha\beta}$ invariant. If the sources are transformed in the same way,

$$\begin{aligned} Z_0 \rho_e &= Z_0 \rho'_e \cos \xi + \rho'_m \sin \xi, & Z_0 \mathbf{J}_e &= Z_0 \mathbf{J}'_e \cos \xi + \mathbf{J}'_m \sin \xi \\ \rho_m &= -Z_0 \rho'_e \sin \xi + \rho'_m \cos \xi, & \mathbf{J}_m &= -Z_0 \mathbf{J}'_e \sin \xi + \mathbf{J}'_m \cos \xi \end{aligned} \tag{6.152}$$

then it is straightforward algebra to show that the generalized Maxwell equations (6.150) are invariant, that is, the equations for the primed fields $(\mathbf{E}', \mathbf{D}', \mathbf{B}', \mathbf{H}')$ are the same as (6.150) with the primed sources present.

The invariance of the equations of electrodynamics under duality transformations shows that it is a matter of convention to speak of a particle possessing an electric charge, but not magnetic charge. The only meaningful question is whether *all* particles have the same ratio of magnetic to electric charge. If they do, then we can make a duality transformation, choosing the angle ξ so that $\rho_m = 0$, $\mathbf{J}_m = 0$. We then have the Maxwell equations as they are usually known.

If, by convention, we choose the electric and magnetic charges of an electron to be $q_e = -e$, $q_m = 0$, then it is known that for a proton, $q_e = +e$ (with the present limits of error being $|q_e(\text{electron}) + q_e(\text{proton})|/e \sim 10^{-20}$) and $|q_m(\text{nucleon})| < 2 \times 10^{-24} Z_0 e$.

This extremely small limit on the magnetic charge of a proton or neutron follows directly from knowing that the average magnetic field at the surface of the earth is not more than 10^{-4} T. The conclusion, to a very high degree of precision, is that the particles of ordinary matter possess only electric charge or, equivalently, they all have the same ratio of magnetic to electric charge. For other, unstable, particles the question of magnetic charge is more open, but no positive evidence exists.

The transformation properties of ρ_m and $\mathbf{J}_m$ under rotations, spatial inversion, and time reversal are important. From the known behavior of **E** and **B** in the usual formulation we deduce from the second line in (6.150) that

ρ_m *is a pseudoscalar density, odd under time reversal*, and
$\mathbf{J}_m$ *is a pseudovector density, even under time reversal.*

Since the symmetries of ρ_m under both spatial inversion and time reversal are opposite to those of ρ_e, it is a necessary consequence of the existence of a particle with both electric and magnetic charges that space inversion and time reversal are no longer valid symmetries of the laws of physics. It is a fact, of course, that

*The presence of the "impedance of free space," $Z_0 = \sqrt{\mu_0/\epsilon_0}$, in the transformation is a consequence of the presence of the dimensionful parameters ϵ_0 and μ_0 in the SI system. Magnetic charge density differs in dimensions from electric charge density in SI units. For users of Gaussian units, put $Z_0 \to 1$.

these symmetry principles are not exactly valid in the realm of elementary particle physics, but present evidence is that their violation is extremely small and associated somehow with the weak interactions. Future developments linking electromagnetic, weak, and perhaps strong, interactions may utilize particles carrying magnetic charge as the vehicle for violation of space inversion and time reversal symmetries. With no evidence for monopoles, this remains speculation.

In spite of the negative evidence for the existence of magnetic monopoles, let us turn to Dirac's ingenious proposal. By considering the quantum mechanics of an electron in the presence of a magnetic monopole, he showed that consistency required the quantization condition,

$$\frac{eg}{4\pi\hbar} = \frac{\alpha g}{Z_0 e} = \frac{n}{2} \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{6.153}$$

where e is the electronic charge, $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine structure constant ($\alpha \approx 1/137$), and g is the magnetic charge of the monopole. The discrete nature of electric charge thus follows from the existence of a monopole. The magnitude of e is not determined, except in terms of the magnetic charge g. The argument can be reversed. With the known value of the fine structure constant, we infer the existence of magnetic monopoles with charges g whose *magnetic "fine structure"* constant is

$$\frac{g^2}{4\pi\mu_0\hbar c} = \frac{n^2}{4}\left(\frac{4\pi\epsilon_0\hbar c}{e^2}\right) \simeq \frac{137}{4}n^2$$

Such monopoles are known as *Dirac monopoles*. Their coupling strength is enormous, making their extraction from matter with dc magnetic fields and their subsequent detection very simple in principle. For instance, the energy loss in matter by a relativistic Dirac monopole is approximately the same as that of a relativistic heavy nucleus with $Z = 137n/2$. It can presumably be distinguished from such a nucleus if it is brought to rest because it will not show an increase in ionization at the end of its range (see Problem 13.11).

6.12 *Discussion of the Dirac Quantization Condition*

Semiclassical considerations can illuminate the Dirac quantization condition (6.153). First, we consider the deflection at large impact parameters of a particle of charge e and mass m by the field of a stationary magnetic monopole of magnetic charge g. At sufficiently large impact parameter, the change in the state of motion of the charged particle can be determined by computing the impulse of the force, assuming the particle is undeflected. The geometry is shown in Fig. 6.6. The particle is incident parallel to the z axis with an impact parameter b and a speed v and is acted on by the radially directed magnetic field of the monopole, $\mathbf{B} = g\mathbf{r}/4\pi r^3$, according to the Lorentz force (6.113). In the approximation that the particle is undeflected, the only force acting throughout the collision is a y component,

$$F_y = evB_x = \frac{eg}{4\pi}\frac{vb}{(b^2 + v^2t^2)^{3/2}} \tag{6.154}$$

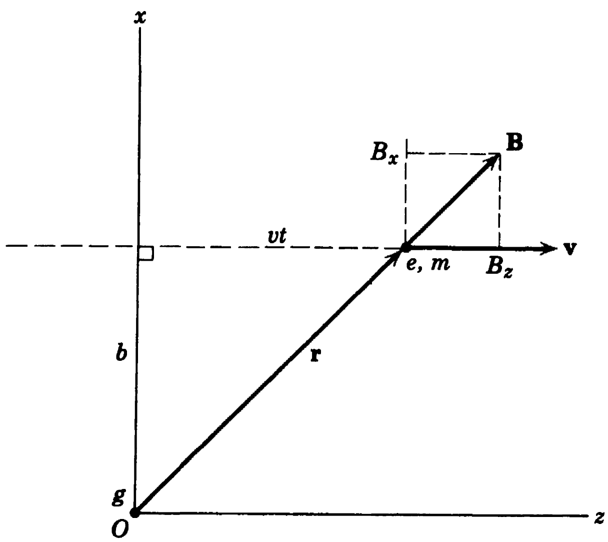

Figure 6.6 Charged particle passing a magnetic monopole at large impact parameter.

The impulse transmitted by this force is

$$\Delta p_y = \frac{egvb}{4\pi}\int_{-\infty}^{\infty}\frac{dt}{(b^2 + v^2t^2)^{3/2}} = \frac{eg}{2\pi b} \tag{6.155}$$

Since the impulse is in the y direction, the particle is deflected out of the plane of Fig. 6.6, that is, in the azimuthal direction. Evidently the particle's angular momentum is changed by the collision, a result that is not surprising in the light of the noncentral nature of the force. The magnitude of the change in angular momentum *is* somewhat surprising, however. There is no z component of $\mathbf{L}$ initially, but there is finally. The change in L_z is

$$\Delta L_z = b\,\Delta p_y = \frac{eg}{2\pi} \tag{6.156}$$

The change in the z component of angular momentum of the particle is independent of the impact parameter b and the speed v of the charged particle. It depends only on the product eg and is a universal value for a charged particle passing a stationary monopole, no matter how far away. If we assume that any change of angular momentum must occur in integral multiples of $\hbar$, we are led immediately to the Dirac quantization condition (6.153).*

The peculiarly universal character of the change in the angular momentum (6.156) of a charged particle in passing a magnetic monopole can be understood by considering the angular momentum contained in the fields of a point electric charge in the presence of a point magnetic monopole. If the monopole g is at $\mathbf{x} = \mathbf{R}$ and the charge e is at $\mathbf{x} = 0$, as indicated in Fig. 6.7, the magnetic and electric fields in all of space are

$$\mathbf{H} = -\frac{g}{4\pi\mu_0}\boldsymbol{\nabla}\left(\frac{1}{r'}\right) = \frac{g}{4\pi\mu_0}\frac{\mathbf{n}'}{r'^2}, \qquad \mathbf{E} = -\frac{e}{4\pi\epsilon_0}\boldsymbol{\nabla}\left(\frac{1}{r}\right) = \frac{e}{4\pi\epsilon_0}\frac{\mathbf{n}}{r^2} \tag{6.157}$$

where $r' = |\mathbf{x} - \mathbf{R}|$, $r = |\mathbf{x}|$, and $\mathbf{n}'$ and $\mathbf{n}$ are unit vectors in the directions of $(\mathbf{x} - \mathbf{R})$ and $\mathbf{x}$, respectively. The angular momentum $\mathbf{L}_{em}$ is given by the volume integral of $\mathbf{x} \times \mathbf{g}$, where $\mathbf{g} = (\mathbf{E} \times \mathbf{H})/c^2$ is the electromagnetic momentum density.

*This argument is essentially due to A. S. Goldhaber, *Phys. Rev.* **140**, B1407 (1965).